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A method for extending planar axis-symmetric parallel manipulators to spatial mechanisms

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ABSTRACT

This paper investigates axis-symmetric parallel manipulators, composed of a central base column and an arm system able to rotate around this column. The arm system includes several actuated upper arms, each connected to a manipulated platform by one or more lower arm linkages. Such manipulators feature an extensive positional workspace in relation to the manipulator footprint and equal manipulator properties in all radial half-planes defined by the common rotation-axis of the upper arms. The similarities between planar manipulators exclusively employing 2-degrees-of-freedom (2-DOF) lower arm linkages and lower mobility spatial manipulators only utilising 5-DOF lower arm linkages are analysed. The 2-DOF linkages are composed of a link with a 1-DOF hinge on both ends whilst the 5-DOF linkages utilise 3-DOF spherical joints and 2-DOF universal joints. By employing a proposed linkage substitution scheme, it is shown how a wide range of spatial axis-symmetric parallel manipulators can be derived from a limited range of planar manipulators of the same type.

1. Introduction

Parallel manipulators (PMs) typically offer several benefits over serial manipulators of the same size. These benefits include higher load capacity, increased acceleration, higher speed and improved position accuracy. However, it is also well known that parallel mechanisms generally suffer from several drawbacks, including highly non-linear manipulator properties and a small workspace in relation to the manipulator footprint.

This paper investigates planar and spatial axis-symmetric PMs composed of a central base column and an arm system capable of infinite rotation around this column. The arm system comprises 2–5 actuated upper arms with a common axis of rotation, 2–7 lower arm linkages (LALs) and a manipulated platform. Such mechanisms are referred to as axis-symmetric parallel manipulators, as they feature equal manipulator properties in all radial half-planes defined by the rotation-axis of the upper arms. The possibility of rotating the entire arm system around the base column leads to an extensive positional workspace in relation to the manipulator footprint and makes it possible to always implement the shortest path between two tool positions.

The analysis in this paper is focused on a subclass of axis-symmetric PMs, for which the LALs are composed of a link with a 3-degrees-of-freedom (3-DOF) spherical joint on one end and a 2-DOF universal joint on the other end. As the link in such a LAL is not susceptible to bending or torsion, it can have a light-weight construction with a high stiffness-to-mass ratio. Using fixed actuators and light-weight lower arm links contributes to low manipulator inertia and a high value of the lowest resonance frequency of the manipulator. By employing six 5-DOF LALs, mechanical over-constraints can be avoided, meaning high-precision assembly of the

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arm system is uncomplicated. Several of the most well-known parallel mechanisms, including the Delta robot [1] and the Gough platform [2], employ similar linkages.

Axis-symmetric PMs exclusively utilising 5-DOF LALs have previously been discussed in several papers and patents [3–11]. In the year 2000, ABB Robotics built and evaluated a two metre tall robot prototype based on this concept. The evaluated prototype, named the SCARA-Tau robot, is a 3-DOF manipulator featuring a positional workspace comparable to that of an IRB 4400 robot. Its linear acceleration is 5 g (2 g), its repeatability is 4 μm (100 μm), its absolute accuracy is 15 μm (500 μm), and its lowest resonance frequency is 30 Hz (10 Hz), where the values within the parentheses are the corresponding values for an IRB 4400 robot.

This paper investigates both planar manipulators and lower mobility spatial manipulators. The planar manipulators are 2T manipulators for two translations and 2T1R manipulators, also featuring a platform rotation. The two translational DOF are in a plane perpendicular to the common axis of rotation of the upper arms and the platform rotation is around an axis parallel to the upper arm axis. The lower mobility spatial manipulators are 3T manipulators for translations in three DOF and 3T1R manipulators for three translations and one rotation. Also here, the platform rotation is around an axis parallel to the common axis of rotation of the upper arms. The described 3T1R motion is a variant of Schoenflies motion and is of great importance in industrial pick-and-place applications and palletising applications.

This paper provides a methodology for deriving axis-symmetric parallel manipulators exclusively employing 5-DOF LALs in these four categories. The proposed methodology is a bottom-up approach, based on a range of axis-symmetric planar PMs exclusively utilising 2-DOF LALs, which are composed of a link with hinges on each end. As the functionality of such manipulators is intuitive, they are used as a starting point for derivation of more complex manipulators. Two variants of a linkage substitution scheme for planar axis-symmetric PMs exclusively utilising 2-DOF LALs are proposed. The proposed schemes allow the derivation of kinematically equivalent planar manipulators exclusively using 5-DOF linkages. The linkage substitution schemes are designed to allow a straightforward extension of the derived planar manipulators to lower mobility spatial mechanisms. By varying the application of the proposed schemes, a large range of different manipulators may be derived from each elementary planar mechanism. It is shown how the described methodology can be used to derive all previously proposed lower mobility spatial manipulators of the discussed type in addition to several novel mechanisms.

The remainder of this paper is organised as follows: Section 2 introduces the joints used in the proposed designs and provides a discussion on parasitic rotation, which is a drawback of some of the discussed manipulators. Section 3 introduces a class of planar manipulators utilising only 2-DOF LALs. All subsequently introduced mechanisms are derived from these manipulators. Thereafter, Section 4 introduces the proposed linkage substitution schemes whilst Section 5 presents examples of derived planar manipulators exclusively using 5-DOF LALs. In Section 6, the derived planar manipulators are extended to lower mobility spatial manipulators whilst the final section summarises the results and provides suggestions for future work. In addition, the paper includes an appendix, where the theory of reciprocal screws is employed to prove the core assumption of the proposed linkage substitution schemes.

2. Axis-symmetric parallel manipulators

2.1. Mobility of the lower arm linkages

The LALs are classified here according to their DOF when potential actuated telescopic links are locked. With two exceptions, the manipulators studied in this paper either exclusively use 2-DOF LALs or 5-DOF LALs.

The 2-DOF LALs consist of a link with a 1-DOF rotational joint (hinge) on each end. Fig. 1(a) illustrates one of these joints. The joint axes of all rotational joints are parallel to the common axis of rotation of the actuated upper arms.

The 5-DOF LALs are composed of a link with a 3-DOF joint on one end and a 2-DOF joint on the other end. The rotation-axes of all joints discussed in this paper are concurrent, which simplifies the kinematics. As the 3-DOF joints are assumed to be heavier than the 2-DOF joints, mounting them closer to the base column reduces the inertia of the actuated arm system. Hence, the 3-DOF joints connect the upper arms and the lower arm links whilst the 2-DOF joints connect the lower arm links and the manipulated platform. It is also possible to use 3-DOF joints on both ends, for example exclusively utilising ball-and-socket joints.

Two types of 3-DOF joints are used, both featuring three concurrent joint axes as shown in Fig. 1(b) and (c). Both types are kinematically equivalent to spherical joints. The first rotation-axis is parallel to the common rotation-axis of the upper arms.

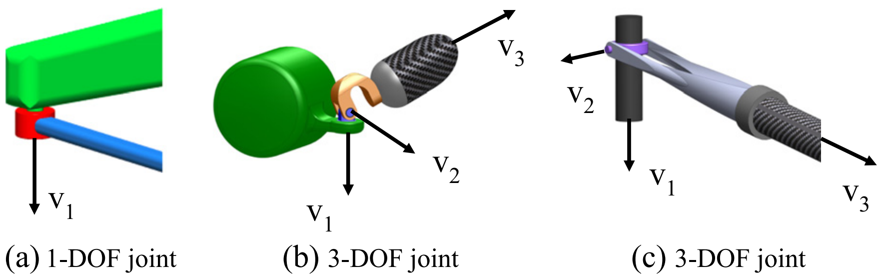


Fig. 1. The 1-DOF and 3-DOF joints employed by the studied manipulators. (a)1-DOF rotational joints. (b)–(c) Two types of 3-DOF joints. The joint axes (v_1, v_2, v_3) are marked in the illustrations.

The axis between the intersection point of the three joint axes of an upper arm joint and the intersection point of the two axes of the corresponding platform joint is here called the LAL axis. The second rotation-axis of the 3-DOF joints is perpendicular to both the LAL axis and the first rotation-axis whilst the third rotation-axis coincides with the LAL axis. The reason for using these joints instead of ball-and-socket joints is to achieve larger working ranges of the joints.

The 2-DOF platform joints are identical to the 3-DOF joints in Fig. 1(b) and (c) except that they do not provide rotation around v_3.

2.2. Parasitic platform rotation

For 2T axis-symmetric PMs, it is typically ideal that the orientation of the tool platform remains constant during radial motion of the tool centre point (TCP). Similarly, for 3T axis-symmetric PMs, it is typically preferable that the tool platform remains constant during both radial and vertical translations of the TCP. We here define a cylindrically constant platform orientation according to:

Definition 1. An axis-symmetric PM exhibits a cylindrically constant platform orientation if it is possible to define an orthogonal coordinate system fixed relative to the manipulated platform, such that in the entire workspace one axis of the defined coordinate system is always parallel to the common axis of rotation of the upper arms and another of the coordinate system axes is always intersecting the common axis of rotation at a 90-degree angle.

Examples of 2T axis-symmetric PMs with cylindrically constant platform orientation are provided in this paper whilst 3T axis-symmetric PMs with this property are proposed in [12,13]. However, neither 2T and 3T manipulators exclusively utilising the described 5-DOF linkages nor 2T manipulators using only the described 2-DOF linkages feature a cylindrically constant platform orientation. The deviation from a cylindrically constant platform orientation is here referred to as parasitic platform rotation. Note that also for a mechanism exhibiting a cylindrically constant platform orientation, only certain choices of TCP avoid parasitic rotation of the tool.

3. Planar manipulators using only 2-DOF lower arm linkages

3.1. Introduction

For the planar mechanisms discussed in this section, the TCP is restricted to a plane perpendicular to the common axis of rotation of the actuated upper arms. The upper arms, lower arm links and platform sections with the potential to collide can typically be arranged to operate in different planes parallel with the TCP-plane. Hence, the risk of collisions between upper arms and lower arm links and between lower arm links and the manipulated platform can be minimised. An axis-symmetric planar PM of this type features an extensive positional workspace in the shape of a disc with a hole in its centre.

The focus here is on planar manipulators exclusively utilising 2-DOF LALs of the type described in Section 2.1. As such manipulators are relatively intuitive, they are a good starting point to derive more complex axis-symmetric PMs. It will be shown that it is possible to derive kinematically equivalent planar mechanisms where all the 2-DOF linkages are replaced with 5-DOF linkages. Additionally, it will be demonstrated how the derived mechanisms can be extended to lower mobility spatial manipulators.

3.2. 2-DOF planar manipulators

Fig. 2 illustrates six 2T axis-symmetric planar PMs utilising two actuated rotating upper arms. The most elementary manipulator of this type is shown in Fig. 2(a). Each of the two 2-DOF LALs imposes only one constraint on the three planar DOF of the manipulated platform. Hence, only the planar position of the platform is constrained, whilst its rotation is unconstrained. Manipulators of this type could still be useful for applications using an axis-symmetric tool, including laser cutting and water jet cutting. The manipulator in Fig. 2(a) is also a building block for more complex axis-symmetric PMs, including the 4-DOF Dual4 mechanism [15] and the 4-DOF Mitsubishi Twin-SCARA device available on the Mitsubishi home-page. Although the latter manipulator is not designed with a common axis of rotation for the actuated upper arms, it could readily be modified in this way.

For the absolute majority of potential applications, the manipulated platform should be fully constrained when all actuators are locked. Using only 2-DOF LALs, this means that three such linkages must be included in the mechanism. Three possibilities to mount these linkages are shown in Fig. 2(b)–(d). In all three cases, the three planar DOF are constrained when the two actuators are locked. However, as only two actuators are used, it is not possible to actuate these DOF independently. Two out of three planar DOF can be fully controlled, whilst the third will vary dependent on the other two. The platform rotation during a radial motion is here viewed as a parasitic rotation.

For the manipulator in Fig. 2(b), the platform rotation is dependent only on the angle of the upper arm where the parallelogram is attached. The manipulator in Fig. 2(c) is an improvement of this design, which reduces the parasitic rotation of the platform. A triangular configuration of the LALs was introduced in [17] for spatial manipulators and was further analysed in [5]. All Duorot mechanisms in Fig. 2(b)–(d) exhibit varying degrees of parasitic platform rotation. The parasitic rotation is only affected by the arrangement of the cluster of two links. It was shown in [14] that the link arrangement minimising such motion is always quadrilateral, where the quadrilateral shape is formed by two LALs, a section of an upper arm and the manipulated platform, as exemplified in Fig. 2(d).

The main drawback of exclusively utilising 2-DOF LALs for 2T axis-symmetric planar PMs is that such manipulators cannot be designed with a cylindrically constant platform orientation according to Def. 1. Whilst it is possible to reduce the parasitic rotations

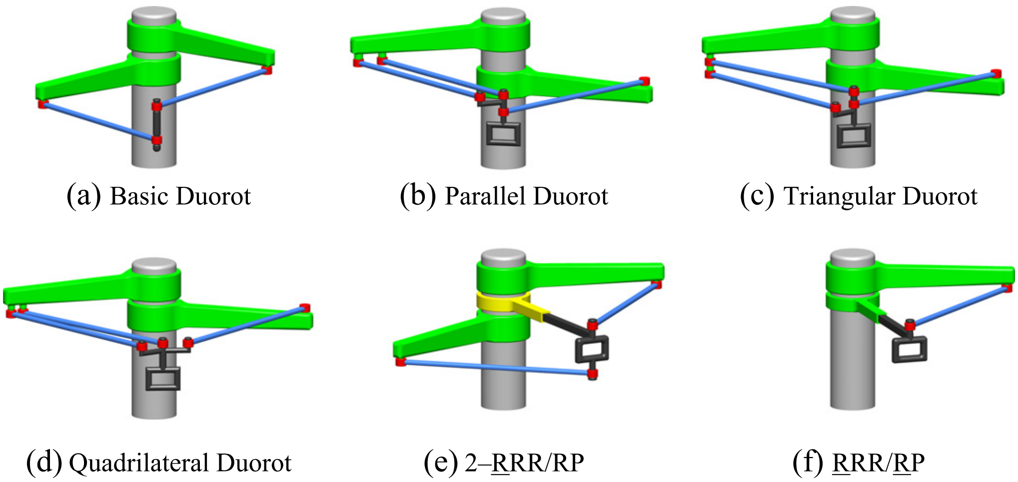


Fig. 2. Six 2-DOF axis-symmetric planar parallel manipulators. Each manipulator consists of a cylindrical base column (grey), two actuated upper arms (green), lower arm links (blue), passive rotational joints (red) and a platform structure (black). The manipulator (e) also includes a passive upper arm (yellow) whilst both (e) and (f) include a passive prismatic joint.

to a low level [14], such motion cannot be completely eliminated. However, as exemplified in Fig. 2(e) and (f), this can be remedied by utilising other types of LALs. If the TCP is chosen to be in the plane that includes the common axis of rotation of the upper arms and is parallel to the prismatic joint, these manipulators do not exhibit parasitic rotation. The manipulator in Fig. 2(e) utilises a passive RP kinematic chain between the base column and the manipulated platform. This kinematic chain constrains the platform orientation without imposing any constraints on the platform position. Using a passive kinematic chain has previously been used in the Tricept manipulator [16]. Fig. 2(f) illustrates a similar variant with reduced inertia, which is achieved by removing one of the two actuated RRR kinematic chains and instead actuating the passive upper arm joint in the RP chain. Although the designs in Fig. 2(e) and (f) have the advantage of a cylindrically constant platform orientation, mechanisms of this type cannot be designed exclusively using 2-DOF linkages or exclusively using 5-DOF linkages and are not further studied in this paper.

3.3. 3-DOF and 4-DOF planar manipulators

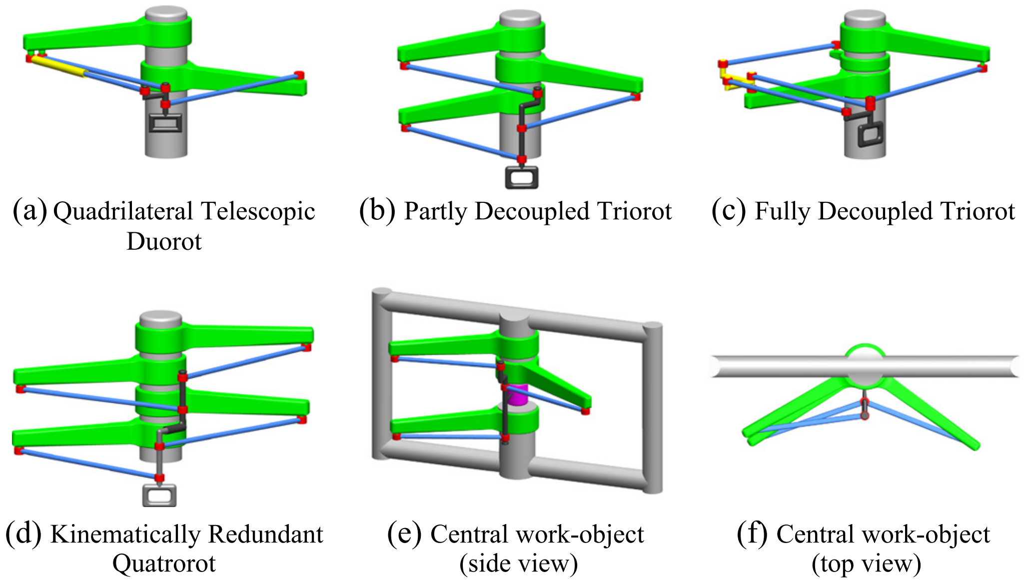
Fig. 3 shows a range of planar axis-symmetric PMs able to manipulate the platform position in two DOF and the platform rotation around an axis parallel to the common axis of rotation of the upper arms. One of the described mechanisms can achieve the same tool configuration with infinite configurations of the actuated arms and is therefore classified as a 4-DOF mechanism whilst all other manipulators in this section are 3-DOF manipulators. The manipulators in Fig. 3(b) and (d) were originally introduced in [14].

As exemplified in Fig. 3(a), the 2-DOF Duorot manipulators in Fig. 2(b)–(d) can be extended to three DOF by replacing one of the fixed-length lower arm links in the cluster of two links with an actuated telescopic link. Because the telescopic link also compensates for the parasitic platform rotation, it is advantageous to base a mechanism of this type on the Quadrilateral Duorot, as it exhibits the minimum parasitic platform rotation of these variants. In this case, a prismatic actuator with shorter stroke length may be used. As one kinematic chain of the resulting mechanism includes two actuators, it is no longer a fully parallel manipulator.

Planar axis-symmetric parallel manipulators employing three actuated rotating upper arms were in [14] named Triorot manipulators. Each upper arm is connected to a manipulated platform by one 2-DOF LAL. The Partly Decoupled Triorot shown in Fig. 3(b) is a special case of this arrangement, for which two of the platform joints have the same planar position but different height. Using this arrangement, the platform position is controlled by two kinematic chains and the third kinematic chain only controls the platform rotation. It is apparent that this manipulator exhibits type two singularities [19] when the uppermost lower arm link is parallel to the horizontal section of the platform, that is, twice per 360° rotation of the platform. All other Triorot manipulators also exhibit two type two singularities per 360° rotation of the platform.

For the Fully Decoupled Triorot in Fig. 3(c), the position and orientation of the manipulated platform are fully decoupled. This is achieved by utilising a parallelogram with two actuated DOF in series with a passive parallelogram. Rotating the middle upper arm modifies the orientation of the intermediate platform section (yellow), which in turn changes the orientation of the tool platform (black), whilst its centre remains in the same position. The platform orientation remains constant if only the upper arms controlling the planar position are rotated, which is not the case for the Partly Decoupled Triorot. If the middle upper arm in Fig. 3(c) is instead rigidly connected to the base column, the resulting manipulator becomes a 2-DOF planar PM exhibiting a constant platform orientation. However, such a design is not axis-symmetric as it features varying manipulator properties in different directions perpendicular to the common axis of rotation of the upper arms. Even if the possibility for infinite rotation is lost, such a design could still be useful. A manipulator of this type would have some similarities to the 2-DOF PacDrive D2 manipulator manufactured by Elau.

Although the workspace to footprint ratio of a Triorot manipulator is extensive compared to most planar PMs, the range of platform rotation is still limited by the two type two singularities occurring during platform rotation. By introducing an additional



(a) Quadrilateral Telescopic Duorot (b) Partly Decoupled Triorot (c) Fully Decoupled Triorot

(d) Kinematically Redundant Quatrorot (e) Central work-object (side view) (f) Central work-object (top view)

Fig. 3. 3-DOF and 4-DOF axis-symmetric planar parallel manipulators exclusively using 2-DOF lower arm linkages. Each manipulator consists of a cylindrical base column (grey), actuated upper arms (green), lower arm links (blue), passive rotational joints (red) and a platform structure (black). The manipulator (a) also includes an actuated telescopic link (yellow) whilst (c) includes an intermediate platform section (yellow). The purple section in (e) illustrates a centrally located work-object.

identical kinematic chain attached to the upper section of the platform, the type two singularities are eliminated and infinite platform rotation is possible in most positions. The mobility of such a manipulator remains three. The drawback of a redundantly actuated manipulator of this type is that if the actuators are operated independently, stress is induced in the mechanism, meaning any modelling inaccuracies could stop the upper arms from reaching their programmed rotation angles. However, redundantly actuated manipulators of this type can be controlled by using force control for one of the actuated upper arms. A manipulator of this type, not designed for infinite rotation, has successfully been used in the haptic device proposed in [18] and is manufactured by Quanser. Fig. 3(d) shows a variant of this mechanism, where a passive prismatic joint has been introduced in the horizontal section of the crank-shaped platform. This modification means that the four actuated upper arms can be operated independently, without causing stress in the mechanism. The proposed manipulator is a kinematically redundant 4-DOF PM with the possibility of infinite platform rotation. Another feature of the manipulator in Fig. 3(d) is that the additional DOF can be used to modify the length of the telescopic link, thereby introducing a variable transmission ratio for the control of the platform rotation.

The illustrations in Fig. 3(e) and (f) show an unorthodox design of a 3-DOF planar axis-symmetric PM with a centrally positioned work-object. The proposed mechanism is derived from Fig. 3(b) and similar mechanisms may possibly be designed based on other axis-symmetric PMs. The proposed design allows the manipulator tool to approach the work-object from all radial directions. There are two advantages of this solution compared to using any of the previously described 3-DOF axis-symmetric planar PMs and mounting the work-object on a rotating table. The first advantage is that the number of actuators required is reduced by one. The second advantage is that the accuracy would be improved as all errors due to the 1-DOF manipulator would be eliminated. The achievable positional workspace is limited by type two singularities [19], when the lower arm links of the two lowest upper arms are parallel. The rotational working range is approximately $\pm 90^\circ$ in most positions. The limitation for additional rotation is type two singularities, which occur when the platform section perpendicular to the common axis of rotation of the upper arms is parallel to the lower arm link of the uppermost actuated upper arm.

4. Linkage substitution schemes

The planar manipulators in the previous section possess the advantage of an uncomplicated design using only low-cost hinges. All except two of the illustrated manipulators exclusively utilise 2-DOF LALs. As a 2-DOF LAL imposes four constraints on a manipulated platform, such manipulators are mechanically over-constrained. These over-constraints can be eliminated by replacing all except one of the 2-DOF LALs with 5-DOF LALs of the type described previously.

A drawback of employing 2-DOF LALs is that the participating links are susceptible to bending and torsion. Typical applications involve tool forces parallel to the common axis of rotation (e.g. gravity). Such forces lead to bending and torsion in the links and joints of the LALs, which must be dimensioned accordingly.

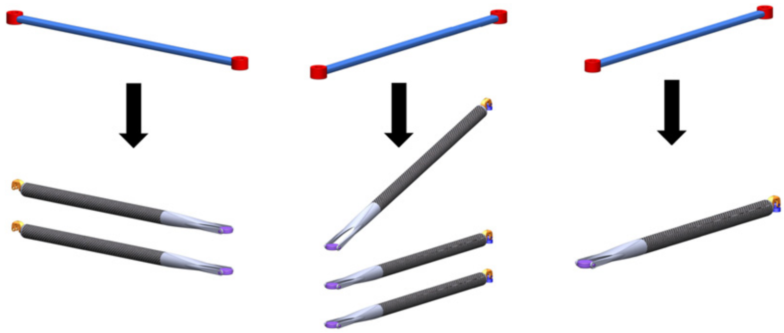
It is possible to maintain equal kinematic properties of a manipulator whilst replacing all 2-DOF LALs by 5-DOF LALs. In this case, all lower arm links can have a light-weight construction with a high stiffness to mass ratio. The drawback is that a minimum of six 5-DOF LALs are required and that more expensive higher order joints must be used. In this section, we propose a linkage substitution scheme valid for all planar PMs exclusively utilising the previously described passive 2-DOF LALs. These manipulators include all mechanisms in Figs. 2 and 3 except the manipulators in Fig. 2(e) and (f). The proposed substitution scheme is not limited to axis-symmetric planar PMs but is valid for all planar PMs where the kinematic chains include a linkage composed of a fixed-length link with passive hinges on both ends. Such manipulators include the 3–RRR and 3–PRR manipulators often analysed in academic literature.

There are several possibilities to replace all 2-DOF LALs in order to achieve a kinematically equivalent planar mechanism exclusively using 5-DOF LALs. The focus here is on linkage substitution schemes allowing a straightforward extension of the derived planar PMs to lower mobility spatial PMs without introducing additional parasitic motion of the manipulated platform. Two such schemes are possible; the first variant is illustrated in Fig. 4(a)–(c) whilst the second possibility is shown in Fig. 4(d)–(g).

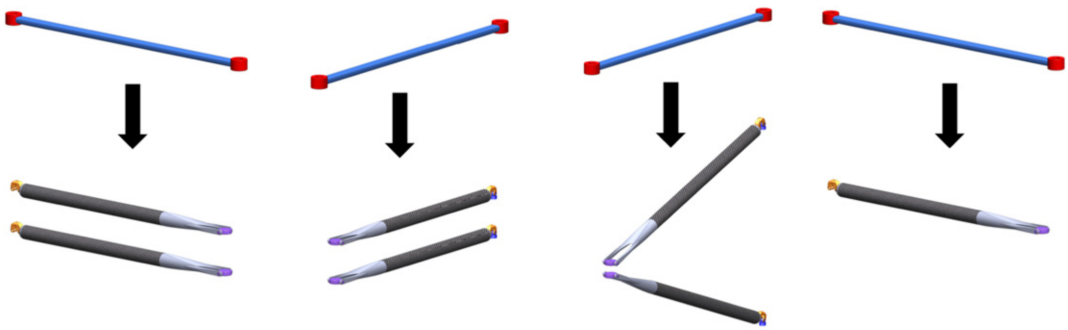
The core idea of both substitution schemes is the assumption that two vertical parallelograms of 5-DOF LALs cause the roll and pitch angles of the manipulated platform to be constant in the entire workspace, meaning the only possible platform rotation is around an axis parallel to the common axis of rotation of the upper arms. This is true regardless of how the parallelograms are angled relative to the horizontal plane and regardless of the angle between the planes formed by the two parallelograms. The exceptions are workspace configurations where the two planes containing the vertical parallelograms are parallel. In such constraint singularities, the platform gains one or more DOF. This core assumption is proved in the appendix by utilising the theory of reciprocal screws.

The first linkage substitution scheme is illustrated in Fig. 4(a)–(c). One 2-DOF LAL is replaced by two 5-DOF LALs mounted in a parallelogram in a plane parallel to the common rotation-axis of the upper arms. In order to simplify the descriptions, such parallelograms are referred to as vertical parallelograms, although that is only true if the common axis of rotation is vertical. A second 2-DOF LAL is replaced with three 5-DOF LALs. Two of these linkages are mounted in a second vertical parallelogram and the third linkage is mounted in the same plane as this parallelogram; however, not parallel to the linkages in the parallelogram. Any additional 2-DOF LALs are replaced by single 5-DOF LALs. The length of the projection of each link cluster in a horizontal plane should equal the length of the 2-DOF LAL the cluster is replacing. The described linkage substitution scheme requires that both the first two substitutions are performed. The remaining linkages are substituted according to Fig. 4(c). These substitutions do not affect the kinematics, but are required to achieve a mechanism that is not over-constrained.

As proved in the appendix, as long as singular configurations are avoided, the pair of vertical parallelograms introduced by the first two substitutions maintain constant roll and pitch angles of the manipulated platform in the entire workspace. Therefore, the described mounting of the third linkage in the cluster of three LALs stops motion parallel to the common axis of rotation of the upper arms, meaning the manipulated platform is only free to move in a plane perpendicular to this axis. Hence, after the first two substitutions, the possible motion of the manipulated platform when employing five 5-DOF LALs is identical to the possible motion when employing two 2-DOF linkages.



(a) 1 - First linkage (b) 1 - Second linkage (c) 1 - Remaining linkages



(d) 2 - First linkage (e) 2 - Second linkage (f) 2 - Third linkage (g) 2 - Remaining linkages

Fig. 4. Linkage substitution scheme for replacing all 2-DOF lower arm linkages with 5-DOF lower arm linkages whilst maintaining the same kinematic properties. The first method is shown in (a)–(c) and the second method in (d)–(g).

With the exception that a 2-DOF linkage including a telescopic link, as in Fig. 3(a), must be replaced by a single 5-DOF linkage including a telescopic link, the substitution scheme does not place any restrictions on which 2-DOF linkage that is substituted for which cluster of 5-DOF linkages. From a kinematic viewpoint, this does not affect the result; however, to achieve practically useful mechanisms certain substitutions are preferable. Each upper arm in Figs. 2 and 3 is either a 'right arm' or a 'left arm'. A small angle between the two planes containing the vertical parallelograms leads to reduced symmetry and stiffness of a manipulator. Hence, it is advantageous to apply the linkage substitution scheme in order for the cluster of three linkages and the cluster of two linkages to be mounted on separate upper arms of different arm type.

The angle between the link in a LAL and a horizontal plane passing through the joint axes intersection point of the upper arm joint of the same LAL is referred to as a LAL angle. The proposed linkage substitution scheme does not include any additional conditions on the mounting positions and LAL angles of the linkages in each cluster. However, as will be demonstrated, these choices affect the manipulator properties and the linkage substitution is preferably performed in order to maximise the symmetry and compactness of the resulting manipulators. Hence, the main focus of the presented work is on manipulators where the LAL angles of the two vertical parallelograms are equal in the entire workspace.

The second possible linkage substitution scheme is shown in Fig. 4(d)–(g). In this case, each of the first two links are substituted for a vertical parallelogram, whilst the third link is substituted for a triangular arrangement of two 5-DOF LALs and any additional 2-DOF linkages are substituted for single 5-DOF linkages. The triangular arrangement locks the platform height to the same horizontal plane. Hence, this scheme requires that at least three 2-DOF LALs are substituted, meaning it is not applicable to the Basic Duorot manipulator in Fig. 2(a).

5. Planar manipulators using only 5-DOF lower arm linkages

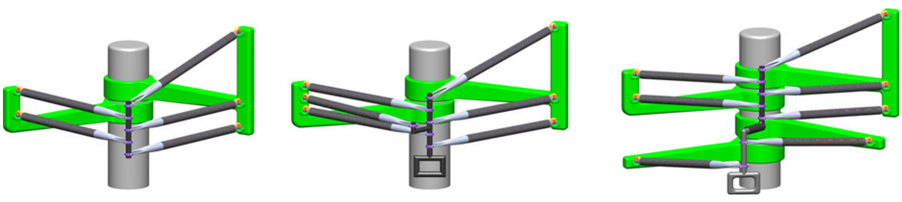
Fig. 5 demonstrates three planar mechanisms derived using the linkage substitution scheme illustrated in Fig. 4(a)–(c). The manipulator in Fig. 5(a) is a variant of the Basic Duorot in Fig. 2(a), the manipulator in Fig. 5(b) is a variant of the Triangular Duorot in Fig. 2(c), whilst the manipulator in Fig. 5(c) is a variant of the Kinematically Redundant Quatrorot in Fig. 3(d).

If the third linkage in the cluster of three LALs is not mounted in the same vertical plane as the parallelogram, the manipulated platform exhibits a parasitic vertical motion when the platform is moved horizontally.

For all Duorot variants except the Basic Duorot, there are three possibilities to distribute the six LALs on the two upper arms. The linkages may be grouped as 3/3, 4/2 or 5/1. These variants may be derived using the linkage substitution schemes by replacing different 2-DOF LALs with different groups of 5-DOF LALs. As shown in the appendix, configurations where the two planes containing a vertical parallelogram are parallel mark a constraint singularity of these manipulators. Hence, for a Parallel Duorot, a 5/1 clustering is not useful. Whilst a 5/1 clustering is generally possible, it typically leads to a small angle between the planes containing the vertical parallelograms, which means the stiffness for platform roll must be carefully evaluated. The angle between the two vertical planes can be increased by using a wider manipulated platform or by basing these manipulator variants on the Intersecting Duorot proposed in [14].

As discussed in Section 4, whilst the two LALs in the same parallelogram must have equal LAL angles, it is possible to use different LAL angles for all other LALs. Hence, it is possible to optimise the manipulator properties for different applications. For example, if the manipulator performs a task on objects below the horizontal motion plane of the tool platform, the LALs can be angled in order for the manipulated platform to be the lowest moving section of the mechanism. The LAL angles may also be selected to maximise the manipulator stiffness. It is advantageous for the platform angle between the parallelogram and the third linkage in the cluster of three to be close to 90°. However, increasing this angle means that the height of the upper arm including these three LALs must be extended.

As discussed previously, the two vertical parallelograms maintain the platform roll and pitch angles when the platform is moved vertically and horizontally. Hence, this linkage configuration is useful for both planar and lower mobility spatial mechanisms. However, for planar mechanisms, where the manipulated platform remains in the same horizontal plane, the linkages in the cluster of three and in the cluster of two are not required to include two parallelograms and other variants may generate more rigid mechanisms.



(a) Basic Duorot (b) Triangular Duorot (c) Kinematically Redundant Quatrorot

Fig. 5. Examples of planar parallel manipulators exclusively utilising 5-DOF lower arm linkages.

6. Spatial manipulators using only 5-DOF lower arm linkages

This section provides a derivation of lower mobility axis-symmetric PMs for 3T motion and 3T1R motion. For the 3T1R manipulators, the rotation-axis of the manipulated platform is parallel to the common axis of rotation of the upper arms.

The studied 3T manipulators include three actuated upper arms or two such arms and one actuated telescopic lower arm link. To constrain all six DOF of the manipulated platform when the actuators are locked, a minimum of six 5-DOF LALs connecting the upper arms and the platform are required. If these six linkages are mounted so at least one linkage is connected to each upper arm but otherwise arbitrarily, the manipulated platform may exhibit coupled parasitic rotations in three DOF. Such manipulators would not be practically useful. The authors believe it is impossible to design 3T axis-symmetric PMs with cylindrically constant platform orientation (Def. 1) employing six of the described 5-DOF linkages. Whilst parasitic motion can be reduced to a low level in only one DOF, it cannot be eliminated. In contrast, 3T1R manipulators of the discussed type are readily designed without parasitic motion, as the additional actuated DOF equals the parasitic DOF for 3T manipulators.

For the studied 3T and 3T1R manipulators, platform rotation around any axis in a plane perpendicular to the common axis of rotation of the upper arms should always be impossible, meaning two tilt angles of the platform should remain constant in the entire workspace. As the planar axis-symmetric manipulators in Section 3 fulfil this criterion, this design objective can be achieved by basing the design on such manipulators. The proposed methodology for kinematic synthesis leads to manipulator variants where all 3T manipulators exhibit parasitic rotation in one DOF and all 3T1R manipulators are free of such motion.

6.1. Replacing a fixed-length link by an actuated telescopic link

As exemplified in Fig. 5, it is possible to derive variants of all manipulators in Figs. 2 and 3 exclusively utilising 2-DOF LALs, where all 2-DOF LALs are replaced by 5-DOF LALs. If the linkage substitution scheme in Fig. 4(a)–(c) is employed, the linkage outside the parallelogram in the cluster of three LALs constrains the platform motion parallel to the common axis of rotation of the actuated upper arms and it is therefore possible to introduce actuation of this positional DOF by replacing the fixed-length link in this LAL with a telescopic link. In case of utilising the scheme in Fig. 4(d)–(g), modifying the length of any of the two linkages in the triangular arrangement in Fig. 4(f) modifies the platform height. With the one exception mentioned in Section 4, the two linkage substitution schemes do not regulate which 2-DOF linkage that is substituted for which cluster of 5-DOF LALs. Hence, an extensive number of different spatial mechanisms may be derived from each of the planar manipulators in Section 3.

The 3-DOF manipulator in Fig. 6(a) is derived from the Parallel Duorot manipulator in Fig. 2. To be classified as a spatial Parallel Duorot, the projection of the cluster of three LALs in a horizontal plane must in each reachable TCP position form a parallelogram. This is only the case if these linkages have equal LAL angles (i.e. are parallel). Otherwise, the projected shape changes during a vertical motion of the TCP and the mechanism is instead classified as a Spatial Quadrilateral Duorot.

Parallel manipulators featuring three translational DOF and one rotational DOF may be designed using two actuated upper arms, four fixed-length lower arm links and two actuated telescopic lower arm links. The mechanism in Fig. 6(b) may be derived from a Triangular Telescopic Duorot, which is a triangular variant of the Quadrilateral Telescopic Duorot in Fig. 3(a).

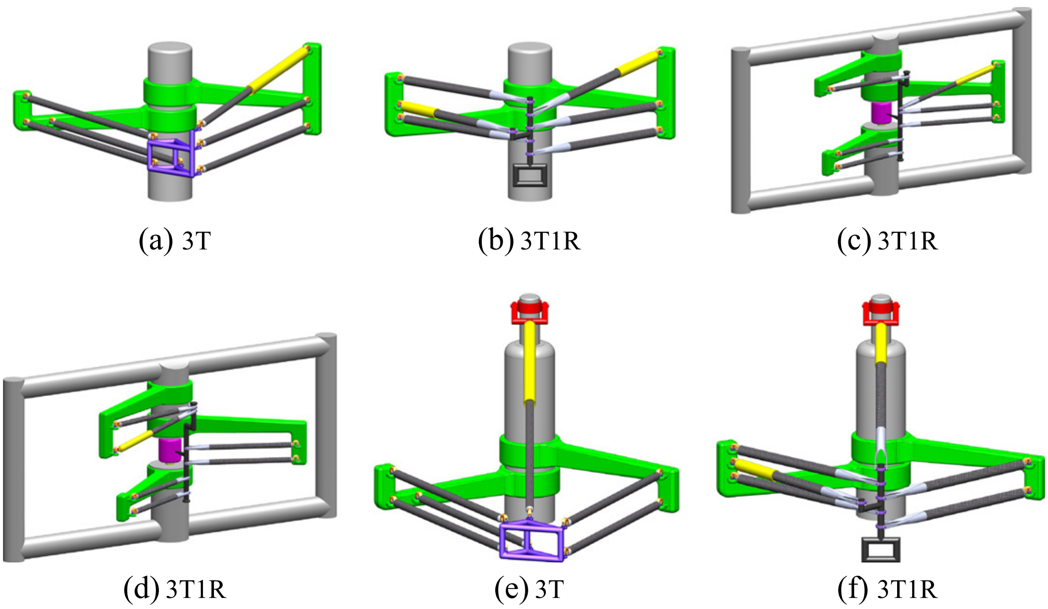


Fig. 6. Spatial 3T and 3T1R axis-symmetric parallel manipulators. The upper arms (green) and the telescopic link (black/yellow) are actuated whilst all other joints are passive.

The 4-DOF manipulator in Fig. 6(c) is derived from the planar mechanism in Fig. 3(e) by employing the linkage substitution scheme in Fig. 4(a)–(c) whilst the more symmetrical variant in Fig. 6(d) is derived by applying the scheme in Fig. 4(d)–(g).

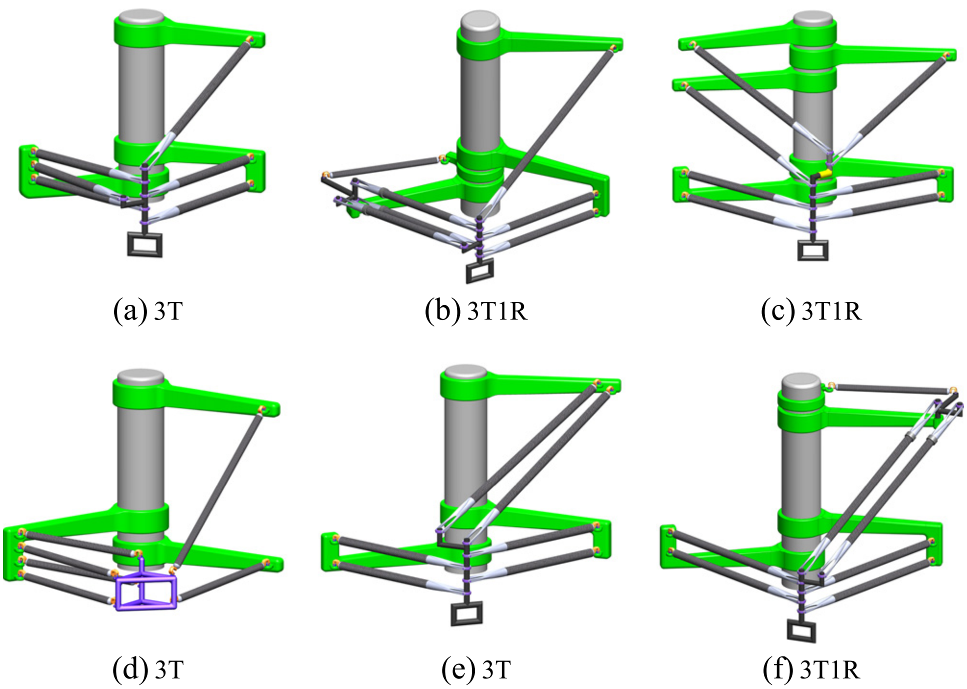
Spatial manipulators similar to those in Fig. 6 may be designed based on all axis-symmetric planar PMs exclusively utilising 2-DOF LALs. The LALs may be clustered 3/3, 4/2 or 5/1. However, according to the discussion in Section 5, the stiffness of a 5/1 clustering must be carefully evaluated. Furthermore, using different LAL angles of the two parallelograms should be avoided for a 5/1 clustering. The reason is that it leads to a variation of the angle between the planes containing the vertical parallelograms when the platform is moved vertically, making it difficult to achieve high rigidity in the entire workspace.

For the manipulators in Fig. 6(a) and (b), the telescopic linkage in the cluster of three LALs is mounted in the same plane as the other two linkages in the same cluster. This is a result of deriving these mechanisms from planar manipulators, where such an arrangement is required to avoid vertical parasitic motion of the platform. However, for the derived spatial manipulators, the endpoints of the telescopic linkage may be repositioned. The advantage of mounting the three linkages in the same plane is a more decoupled mechanism, as the telescopically actuated link does not have to compensate for motions of the two upper arms in order for the manipulated platform to remain in the same horizontal plane. The drawback of this arrangement is that in order to achieve a large angle between the linkages in the parallelogram and the telescopic link, the height of the upper arms must be large, which leads to increased mass of these arms. As demonstrated in Fig. 6(e) and (f), this drawback can be remedied by moving the 3-DOF joint of the telescopic linkage from an upper arm to the top of the base column. The 3-DOF joint of this linkage is modified compared to the other 3-DOF joints of the mechanism; however, it is still kinematically equivalent to a passive spherical joint. In contrast to the other mechanisms in Fig. 6, the manipulator in Fig. 6(e) is fully parallel.

6.2. Adding an actuated upper arm

The spatial axis-symmetric PMs in this section may be derived from planar axis-symmetric PMs by applying any of the two linkage substitution schemes proposed in Section 4 and then moving the upper arm joint of the 5-DOF LAL constraining the platform height to an additional upper arm. The platform joint of this LAL is not required to remain in the same position. By following this methodology it is possible to derive all manipulators proposed in [7,10,11]. For example, the original SCARA-Tau manipulator [10] may be derived based on a planar Duorot mechanism where the cluster of two LALs forms a parallelogram as in Fig. 2(b); however, with all three platform joints separated as in Fig. 2(d).

For the original SCARA-Tau manipulator, the LALs are clustered in groups of 3, 2 and 1 linkages. Such a 3/2/1 clustering is also employed by the 3-DOF mechanism in Fig. 7(a), which may be derived from the Triangular Duorot manipulator in Fig. 2(c). It employs the triangular link configuration proposed in [17]; however, with five of the platform joints in a collinear arrangement. The advantages of a triangular link configuration are discussed in [5,17].



(a) 3T (b) 3T1R (c) 3T1R

(d) 3T (e) 3T (f) 3T1R

Fig. 7. Spatial 3T and 3T1R axis-symmetric parallel manipulators. The manipulators in (a)–(b) employ a 3/2/1 clustering of the lower arm linkages, the manipulator in (d) employs a 4/1/1 clustering whilst the manipulators in (e)–(f) employ a 2/2/2 clustering.

The 4-DOF manipulator in Fig. 7(b) may be derived from the planar manipulator in Fig. 3(c). All four upper arms (green) are actuated. One upper arm is connected via a 5-DOF linkage to an intermediate platform, which is pivoting on another upper arm. This 5-DOF linkage remains parallel to a horizontal line intersecting the pivot axis and the common axis of rotation of the upper arms. The intermediate platform is attached to the tool platform via three parallel 5-DOF LALs. The actuation of platform rotation is fully decoupled from the actuation of spatial position. The rotation of the tool platform is solely controlled by one upper arm and remains constant when the other three upper arms are rotated. Similarly, the centre of the tool platform remains in the same position when the upper arm controlling the platform rotation is actuated. Actuation of two DOF of a parallelogram has been proposed for other parallel mechanisms [20].

If the upper arm manipulating the rotation of the tool platform is instead rigidly connected to the base cylinder, the resulting manipulator is a 3-DOF PM with constant orientation of the tool platform. Such a mechanism is not an axis-symmetric manipulator, as its properties are not identical in different radial directions. However, a manipulator of this type may still be useful for numerous applications.

The 4-DOF manipulator in Fig. 7(c) is derived by applying the linkage substitution scheme in Fig. 4(a)–(c) to the planar manipulator in Fig. 3(d). The 2-DOF linkage on the lowest upper arm is replaced by the cluster of three 5-DOF linkages, the 2-DOF linkage on the second lowest upper arm is replaced by the cluster of two 5-DOF linkages and the 2-DOF linkages on the other two upper arms are replaced by single 5-DOF linkages. Thereafter, the uppermost 5-DOF linkage in the cluster of three is moved to an additional upper arm. This manipulator allows three platform translations and infinite rotation of the platform around a vertical axis. The inclusion of a passive prismatic joint to separate two platform sections means the mechanism is kinematically redundant instead of redundantly actuated. As long as joint limitations are not reached, the five upper arms may be actuated independently without introducing stress in the mechanism. If the passive prismatic joint is replaced by a passive parallelogram, this manipulator becomes identical to one of the mechanisms proposed in [11].

Performing the two substitutions in Fig. 4(a) and (b) on two 2-DOF LALs on the same upper arm and following this substitution by moving one 5-DOF LAL to an additional actuated upper arm leads to a PM with four LALs on one upper arm. Fig. 7(d) shows an example of such a manipulator, derived from the planar Triangular Duorot in Fig. 2(c). A more symmetric variant where the cluster of four LALs are mounted on the uppermost upper arm is also possible. As discussed in Sections 5 and 6.1, the stiffness of such variants must be carefully evaluated.

The 3-DOF spatial manipulators derived by introducing an additional upper arm always feature a 3/2/1 or a 4/1/1 clustering of the LALs on the upper arms. However, two constant tilt angles of the manipulated platform may also be achieved by utilising a 2/2/2 clustering of the LALs. Such mechanisms are named Symmetric SCARA manipulators [7] and may be derived from any of the previous variants with 3/2/1 clustering of the LALs by moving the linkage in the cluster of three not included in the parallelogram to the upper arm with a single linkage. Fig. 7(e)–(f) show two examples of such manipulators. The 3-DOF manipulator in Fig. 7(e) is a variant of a mechanism proposed in [7] whilst the 4-DOF manipulator in Fig. 7(f) is a variant of the manipulator introduced in Fig. 7(b).

Solutions to the inverse kinematics problem for 3-DOF axis-symmetric parallel manipulators with parasitic motion, applicable to the mechanisms in Fig. 7(a), (d) and (e), are provided in [22].

7. Conclusion and future work

This paper introduced a methodology to derive planar and lower mobility spatial axis-symmetric PMs, for which the lower arm links are only susceptible to axial forces. The proposed design scheme is a bottom up approach, based on a range of elementary planar axis-symmetric PMs exclusively employing 2-DOF LALs. By applying a proposed linkage substitution scheme, kinematically equivalent planar manipulators exclusively using 5-DOF LALs are derived. The proposed linkage substitution scheme allows the derived planar mechanisms to be extended to lower mobility spatial mechanisms without introducing additional parasitic rotations. The proposed methodology may be used to derive all previously proposed 3-DOF and 4-DOF spatial axis-symmetric PMs of the discussed type and was used here to derive several novel mechanisms.

In order to evaluate the practical applicability of the derived manipulators, further research is required. An analysis of the kinetostatic and dynamic properties of these mechanisms would be particularly valuable.

Appendix A. Mobility of a mechanism employing two vertical parallelograms

This appendix provides a kinematic analysis of a moving platform, which is attached to a fixed base by two parallelograms composed of the previously described 5-DOF linkages. We provide necessary and sufficient conditions for constraint singularities of this mechanism, along with analytical expressions of its permissible motion, thereby demonstrating that if singular configurations are avoided, two tilt angles of the platform remain constant.

The forces and torques acting on a rigid body can be described by a six-dimensional vector called a wrench whilst the instantaneous motion of a rigid body can be represented by another six-dimensional vector called a twist. Both twists and wrenches can be represented by screws, which are six-dimensional vectors constructed from pairs of three-dimensional vectors. Two screws are said to be reciprocal to each other if the instantaneous virtual work between a twist and a wrench associated with the two screws is zero. When a screw system of wrenches (wrench system) acts on a rigid body, the reciprocal screw system of twists (twist system) describes the permissible motion of the body. Therefore, the reciprocity property makes it possible to obtain motion information from the corresponding constraint information and vice versa [21].

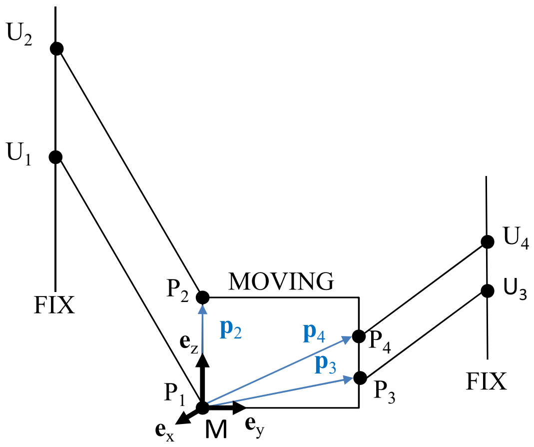


Fig. A.1. The studied mechanism.

The analysed mechanism is illustrated in Fig. A.1. The joints are denoted U_i and P_i. An orthogonal coordinate system **M** with arbitrary orientation is defined with its origin in the joint axes intersection point of P_1. The positions of the joints U_i and P_i in the coordinate system **M** are given by $\mathbf{u}_i$ and $\mathbf{p}_i$, respectively. The mechanism is defined by

$$\|\mathbf{u}_1\| = \|\mathbf{p}_2 - \mathbf{u}_2\|, \tag{A.1}$$

$$\|\mathbf{p}_3 - \mathbf{u}_3\| = \|\mathbf{p}_4 - \mathbf{u}_4\|, \tag{A.2}$$

$$\|\mathbf{u}_2 - \mathbf{u}_1\| = \|\mathbf{p}_2\|, \tag{A.3}$$

$$\|\mathbf{u}_4 - \mathbf{u}_3\| = \|\mathbf{p}_4 - \mathbf{p}_3\|, \tag{A.4}$$

$$\mathbf{p}_4 - \mathbf{p}_3 = \gamma \mathbf{p}_2, \tag{A.5}$$

$$\mathbf{u}_4 - \mathbf{u}_3 = \gamma(\mathbf{u}_2 - \mathbf{u}_1), \tag{A.6}$$

where $\|\mathbf{p}_2\| > 0$ and $\gamma > 0$. We here analyse an arbitrary configuration in which the two links in each link pair are parallel:

$$\mathbf{l}_1 = -\mathbf{u}_1 = \mathbf{p}_2 - \mathbf{u}_2, \tag{A.7}$$

$$\mathbf{l}_2 = \mathbf{p}_3 - \mathbf{u}_3 = \mathbf{p}_4 - \mathbf{u}_4, \tag{A.8}$$

$$\|\mathbf{l}_i\| > 0, i = 1, 2. \tag{A.9}$$

Four wrenches act on the moving platform. The resulting screw system with respect to the coordinate system **M** is

$$\mathbf{A} = \begin{bmatrix} \mathbf{l}_1^{\mathrm{T}} & \mathbf{0} \\ \mathbf{l}_1^{\mathrm{T}} & (\mathbf{p}_2 \times \mathbf{l}_1)^{\mathrm{T}} \\ \mathbf{l}_2^{\mathrm{T}} & (\mathbf{p}_3 \times \mathbf{l}_2)^{\mathrm{T}} \\ \mathbf{l}_2^{\mathrm{T}} & (\mathbf{p}_4 \times \mathbf{l}_2)^{\mathrm{T}} \end{bmatrix}, \tag{A.10}$$

where $\mathbf{A} \in \mathbb{R}^{4\times6}$. Each row of **A** represents a wrench, which does not cause a platform displacement if applied to the platform, i.e. the constraint imposed by the corresponding linkage. As the dimension of the null space of the wrench system determines the degrees of freedom of the mechanism, we will first establish a result regarding the rank of **A**.

For degenerate configurations when $\mathbf{l}_1$ or $\mathbf{l}_2$ are parallel to $\mathbf{p}_2$, it is clear that $\mathbf{A}$ looses rank. In the first case, the first two rows are identical, whilst in the second case, rows three and four are the same. Such configurations correspond to the links in a parallelogram being collinear, which is not physically possible due to collisions. In the following analysis we assume:

$$\mathbf{l}_i \times \mathbf{p}_2 \neq \mathbf{0}, \quad i = 1, 2. \tag{A.11}$$

Let Π_1 and Π_2 denote the two linear subspaces spanned by ($\mathbf{l}_1$, $\mathbf{p}_2$) and ($\mathbf{l}_2$, $\mathbf{p}_2$), respectively. We will show that $\mathbf{A}$ is of full rank if and only if Π_1 and Π_2 do not coincide. We only need to consider the case where $\mathbf{l}_1$ and $\mathbf{l}_2$ are not linearly dependent, as if $\mathbf{l}_1 = c\mathbf{l}_2$ for some $c \neq 0$, then $\Pi_1 = \Pi_2$ follows trivially by their definition. In this case, $\mathbf{A}$ is rank deficient, as the difference between row four and row three is a linear combination of the difference between row two and row one. Our proof will be by contradiction. Assume that $\mathbf{A}$ is rank deficient and that Π_1 and Π_2 do not coincide, then there must exist a $\mathbb{R}^4 \ni \mathbf{z} \neq \mathbf{0}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{z} = \mathbf{0}. \tag{A.12}$$

Consequently

$$z_1 = -z_2, \tag{A.13}$$

$$z_3 = -z_4, \tag{A.14}$$

and

$$z_2(\mathbf{p}_2 \times \mathbf{l}_1) + z_3(\mathbf{p}_3 \times \mathbf{l}_2) + z_4(\mathbf{p}_4 \times \mathbf{l}_2) = z_2(\mathbf{p}_2 \times \mathbf{l}_1) - z_3(\mathbf{p}_4 - \mathbf{p}_3) \times \mathbf{l}_2 = \mathbf{0}. \tag{A.15}$$

Using (A.5) to simplify (A.15) gives us

$$z_2(\mathbf{p}_2 \times \mathbf{l}_1) = z_3\gamma(\mathbf{p}_2 \times \mathbf{l}_2). \tag{A.16}$$

The Eqs. (A.13) and (A.14) and the fact that $\mathbf{z} \neq \mathbf{0}$ mean z_2 and z_3 cannot both be zero. Hence, according to (A.11) and (A.16), none of z_2 and z_3 can be zero. This means that the normals of the subspaces Π_1 and Π_2, given by ($\mathbf{p}_2 \times \mathbf{l}_1$) and ($\mathbf{p}_2 \times \mathbf{l}_2$) respectively, are linearly dependent. Therefore, these linear subspaces must coincide, which then contradicts our initial assumption.

Now to show the reverse implication, if $\Pi_1 = \Pi_2$, we can write $\mathbf{l}_1 = \lambda_1\mathbf{l}_2 + \lambda_2\mathbf{p}_2$. Consider the following product

$$\begin{aligned}\mathbf{A}^{\mathrm{T}}\begin{bmatrix}\gamma \\ -\gamma \\ \lambda_1 \\ -\lambda_1\end{bmatrix} &= \begin{bmatrix}\gamma(\lambda_1\mathbf{l}_2 + \lambda_2\mathbf{p}_2) - \gamma(\lambda_1\mathbf{l}_2 + \lambda_2\mathbf{p}_2) + \lambda_1\mathbf{l}_2 - \lambda_1\mathbf{l}_2 \\ -\gamma(\mathbf{p}_2 \times (\lambda_1\mathbf{l}_2 + \lambda_2\mathbf{p}_2)) + \lambda_1(\mathbf{p}_3 \times \mathbf{l}_2) - \lambda_1(\mathbf{p}_4 \times \mathbf{l}_2)\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{0} \\ -\gamma(\mathbf{p}_2 \times \lambda_1\mathbf{l}_2) - \gamma(\mathbf{p}_2 \times \lambda_2\mathbf{p}_2) - \lambda_1((\mathbf{p}_4 - \mathbf{p}_3) \times \mathbf{l}_2)\end{bmatrix} = \\ &= \begin{bmatrix}\mathbf{0} \\ -\gamma(\mathbf{p}_2 \times \lambda_1\mathbf{l}_2) + \lambda_1(\gamma\mathbf{p}_2 \times \mathbf{l}_2)\end{bmatrix} = \mathbf{0}.\end{aligned} \tag{A.17}$$

As $[\gamma - \gamma\, \lambda_1 - \lambda_1]^{\mathrm{T}} \neq \mathbf{0}$, the rows of $\mathbf{A}$ must be linearly dependent, meaning $\mathbf{A}$ is rank deficient.

We now turn our attention to determining the permissible motion of the mechanism in Fig. A.1 in the non-singular case. It is well known that the freedom in a rigid body configuration is determined by the reciprocal screws to the associated screw system:

$$\mathbf{A}\mathbf{s} = \mathbf{0}. \tag{A.18}$$

Each twist $\mathbf{s} = \begin{bmatrix}\mathbf{v} \\ \mathbf{w}\end{bmatrix} \in \mathbb{R}^6$ is reciprocal to each wrench in $\mathbf{A}$ and represents a permissible displacement of the platform. The reciprocal screw system to an n-screw system of full rank has an order of $6 - n$. In this case, the twists turn out to have a particularly simple form. If $\mathbf{A}$ is of full rank, then the entire space of solutions to (A.18) is spanned by

$$\mathbf{s}_1 = \begin{bmatrix}\mathbf{n}_1 \\ \mathbf{0}\end{bmatrix} \tag{A.19}$$

and

$$\mathbf{s}_2 = \begin{bmatrix}\alpha\mathbf{l}_1 + \beta\mathbf{l}_2 \\ -\mathbf{p}_2\end{bmatrix}, \tag{A.20}$$

where $\mathbf{n}_1$ is any non-zero solution to $\begin{bmatrix}\mathbf{l_1}^T\\ \mathbf{l_2}^T\end{bmatrix}\mathbf{n_1} = \mathbf{0}$, and

$$\alpha = -\frac{\left(\mathbf{l_1}^T\mathbf{l_2}\right)\left((\mathbf{p}_4\times\mathbf{l_2})^T\mathbf{p}_2\right)}{\left(\mathbf{l_1}^T\mathbf{l_1}\right)\left(\mathbf{l_2}^T\mathbf{l_2}\right)-\left(\mathbf{l_1}^T\mathbf{l_2}\right)^2}, \tag{A.21}$$

$$\beta = \frac{\left(\mathbf{l_1}^T\mathbf{l_1}\right)\left((\mathbf{p}_4\times\mathbf{l_2})^T\mathbf{p}_2\right)}{\left(\mathbf{l_1}^T\mathbf{l_1}\right)\left(\mathbf{l_2}^T\mathbf{l_2}\right)-\left(\mathbf{l_1}^T\mathbf{l_2}\right)^{\mathbf{2}}}. \tag{A.22}$$

This result follows trivially as $\mathbf{As}_i = \mathbf{0}$, $||\mathbf{s}_i|| \neq \mathbf{0}$ $(i = 1, 2)$ and $\mathbf{s}_1^T\mathbf{s}_2 = \mathbf{0}$, all easily verified by insertion.

A direct implication of the derived solution space is that, as long as singular configurations are avoided, the permissible motion of the mechanism in Fig. A.1 is composed of a rotation about $\mathbf{p}_2$ and a translation. Hence, unless the subspaces Π_1 and Π_2 coincide (meaning the planes formed by the two vertical parallelograms are parallel), two tilt angles of the moving platform will remain constant.

Note that the assumptions (A.7) and (A.8) specify that the two links in each link pair are parallel. In a singularity, other rotations of the platform are possible and once a rotation around another axis than $\mathbf{p}_2$ has occurred, these assumptions are no longer valid. However, as the mechanism should always avoid such singularities, the motion in a singularity is not further analysed here.

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